

# Positive Linear Maps and Perturbation bounds of Matrices

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**Abstract.** We show how positive unital linear maps can be used to obtain lower bounds for the maximum distance between the eigenvalues of two normal matrices. Some related bounds for the spread and condition number of Hermitian matrices are also discussed here.

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# 1 Introduction

Recently, Bhatia and Sharma [1, 4, 5] have shown that how positive unital linear maps can be used to obtain matrix inequalities. In particular, they have obtained some old and new lower bounds for the spread of a matrix. In this paper we show that their technique can be extended and positive unital linear maps can also be used to study the spectral variations of Hermitian and normal matrices.

Let  $\mathbb{M}(n)$  be the algebra of all  $n \times n$  complex matrices. Let  $\langle x, y \rangle$  be the standard inner product on  $C^n$  defined as  $\langle x, y \rangle = \sum_{i=1}^n \overline{x_i} y_i$ , and let  $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ . The numerical range of an element  $A \in \mathbb{M}(n)$  is the set

$$W(A) = \{\langle x, Ax \rangle : \|x\| = 1\}.$$

The Toeplitz-Hausdorff Theorem [7, 12] says that  $W(A)$  is a convex subset of the complex plane for all  $A \in \mathbb{M}(n)$ . For a normal matrix  $A$ ,

$$W(A) = Co(\sigma(A))$$

where  $Co(\sigma(A))$  denotes the convex hull of the spectrum  $\sigma(A)$  of  $A$ . For non-normal matrices,  $W(A)$  may be bigger than  $Co(\sigma(A))$ . The diameter of  $W(A)$  is defined as

$$\text{diam } W(A) = \max_{i,j} \{|z_i - z_j| : z_i, z_j \in W(A)\}.$$

A linear map  $\Phi : \mathbb{M}(n) \rightarrow \mathbb{M}(k)$  is called positive if  $\Phi(A)$  is positive semidefinite (psd) whenever  $A$  has that property, and unital if  $\Phi(I) = I$ . When  $k = 1$ , such a map is called positive, unital, linear functional and is denoted by the lower case letter  $\varphi$ .

Bhatia and Davis [3] have proved that if  $\Phi$  is any positive unital linear map and the spectrum of any Hermitian matrix  $A$  is contained in the interval  $[m, M]$ , then

$$\Phi(A^2) - \Phi(A)^2 \leq \frac{(M - m)^2}{4}. \quad (1.1)$$

Bhatia and Sharma [4] have extended this for arbitrary matrices. One more extension of (1.1) in the special case when  $A$  is normal and  $\varphi$  is linear functional is given in [6]. They have augmented this technique with another use of positive unital linear maps and showed that if  $\Phi_1$  and  $\Phi_2$  are positive unital linear maps from  $\mathbb{M}(n)$  to  $\mathbb{M}(k)$ , then for every Hermitian matrix  $A \in \mathbb{M}(n)$  we have

$$\|\Phi_1(A) - \Phi_2(A)\| \leq \text{diam } W(A) \quad (1.2)$$

where  $\|\cdot\|$  denotes the spectral norm. Further, if  $\varphi_1$  and  $\varphi_2$  are positive unital linear functionals on  $\mathbb{M}(n)$ , then for every matrix  $A$  in  $\mathbb{M}(n)$

$$|\varphi_1(A) - \varphi_2(A)| \leq \text{diam } W(A). \quad (1.3)$$

For more details, see [5, 6]. Using these inequalities they have derived various old and new bounds for the spread of matrices. In a similar spirit we discuss here perturbation bounds related to inequalities involving positive linear maps.

For an expository review of bounds for the distance between the eigenvalues of two matrices  $A$  and  $B$  in terms of expressions involving  $\|A - B\|$ , see [2]. In the present context an inequality of interest to us is due to Weyl (1911) which says that if  $A$  and  $B$  are Hermitian matrices, then

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\downarrow(B)\| \leq \|A - B\| \leq \|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \quad (1.4)$$

where  $\text{Ei } g^\downarrow(A)$  ( $\text{Ei } g^\uparrow(A)$ ) denotes a diagonal matrix whose diagonal entries are the eigenvalues of  $A$  in decreasing (increasing) order, see [1, 13].

For any two elements  $A$  and  $B$  of  $\mathbb{M}(n)$ , we define

$$s(W(A), W(B)) = \max_{i,j} \{|w_i(A) - w_j(B)| : w_i(A) \in W(A), w_j(B) \in W(B)\}.$$

Note that  $s(W(A), W(B)) = \text{diam } W(A)$  for  $A = B$ .

We show that the inequality (1.3) can be extended for two matrices  $A$  and  $B$  with  $\text{diam } W(A)$  replaced by  $s(W(A), W(B))$ , (see Theorem 2.1, below). In the special case when  $A$  and  $B$  are normal we get the lower bound for the maximum distance between the eigenvalues of  $A$  and  $B$  (Corollary 2.1). Likewise, we obtain an extension of (1.2) for two Hermitian matrices (Theorem 2.2).

## 2 Main results

**Theorem 2.1.** Let  $\varphi_i : \mathbb{M}(n) \rightarrow \mathbb{C}$  be positive unital linear functionals,  $i = 1, 2$ . Let  $A$  and  $B$  be any two elements of  $\mathbb{M}(n)$ . Then

$$|\varphi_1(A) - \varphi_2(B)| \leq s(W(A), W(B)). \quad (2.1)$$

**Proof.** If  $A \in \mathbb{M}(n)$  then every positive unital linear functional  $\varphi(A)$  can be expressed as the convex combination of  $n$  complex numbers, each of which is in the numerical

range of  $A$ , see [5]. Therefore, there exists complex numbers  $z_i(A)$  in  $W(A)$  and  $z_j(B)$  in  $W(B)$  such that

$$\varphi_1(A) = \sum_{i=1}^n \alpha_i z_i(A) \text{ and } \varphi_2(B) = \sum_{j=1}^n \beta_j z_j(B) ,$$

where  $\alpha_i$  and  $\beta_j$  are non-negative real numbers such that  $\sum_{i=1}^n \alpha_i = \sum_{j=1}^n \beta_j = 1$ .

By the Toeplitz-Hausdorff Theorem,  $\varphi_1(A) \in W(A)$  and  $\varphi_2(B) \in W(B)$ , and so (2.1) follows immediately. ■

**Lemma 2.1.** Let  $U$  and  $V$  denote the convex hulls of complex numbers  $z_i(U)$  and  $z_j(V)$  respectively,  $i, j = 1, 2, \dots, n$ . Then, the inequality

$$|u - v| \leq \max_{i,j} \{|z_i(U) - z_j(V)|\} , \quad (2.2)$$

holds true for all complex numbers  $u \in U$  and  $v \in V$ .

**Proof.** Since  $u$  and  $v$  are in the convex hulls of complex numbers  $z_i(U)$  and  $z_j(V)$  respectively, we can write

$$u = \sum_{i=1}^n p_i z_i(U) \text{ and } v = \sum_{j=1}^n q_j z_j(V) ,$$

where  $p_i$  and  $q_j$  are non-negative real numbers such that  $\sum_{i=1}^n p_i = \sum_{j=1}^n q_j = 1$ .

We therefore have

$$\begin{aligned} |u - v| &= \left| \sum_{j=1}^n q_j (u - z_j(V)) \right| \\ &\leq \sum_{j=1}^n q_j |u - z_j(V)| \\ &\leq \max_j |u - z_j(V)| \\ &= \max_j \left| \sum_{i=1}^n p_i (z_i(U) - z_j(V)) \right| \\ &\leq \max_j \sum_{i=1}^n p_i |z_i(U) - z_j(V)| \\ &\leq \max_{i,j} \{|z_i(U) - z_j(V)|\} . \end{aligned}$$

This proves the lemma. ■

**Corollary 2.1.** Under the conditions of Theorem 2.1, if  $A$  and  $B$  are normal matrices, then

$$|\varphi_1(A) - \varphi_2(B)| \leq \max_{i,j} |\lambda_i(A) - \lambda_j(B)|, \quad (2.3)$$

where  $\lambda_i(A)$  and  $\lambda_j(B)$  are the eigenvalues of  $A$  and  $B$ , respectively.

**Proof.** If  $A$  is normal, then numerical range of  $A$  is the convex polygon spanned by the eigenvalues of  $A$ . So,  $W(A)$  and  $W(B)$  are the convex hulls of the eigenvalues  $\lambda_i(A)$  and  $\lambda_j(B)$ , respectively. It follows from above lemma that

$$s(W(A), W(B)) = \max_{i,j} |\lambda_i(A) - \lambda_j(B)|.$$

The assertions of the corollary now follows from Theorem 2.1. ■

**Theorem 2.2.** Let  $\Phi_1$  and  $\Phi_2$  be any two positive unital linear maps from  $\mathbb{M}(n)$  into  $\mathbb{M}(k)$ . Let  $A$  and  $B$  be any two Hermitian elements of  $\mathbb{M}(n)$ . Then

$$\|\Phi_1(A) - \Phi_2(B)\| \leq \|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\|. \quad (2.4)$$

**Proof.** If  $A$  is an  $n \times n$  Hermitian matrix then  $\lambda_n^\downarrow(A)I \leq A \leq \lambda_1^\downarrow(A)I$ . The linear map  $\Phi_1$  preserves order and take the identity  $I$  in  $\mathbb{M}(n)$  to  $I$  in  $\mathbb{M}(k)$ . So we have  $\lambda_n^\downarrow(A)I \leq \Phi_1(A) \leq \lambda_1^\downarrow(A)I$ . Likewise, we have  $\lambda_1^\uparrow(B)I \leq \Phi_2(B) \leq \lambda_n^\uparrow(B)I$ . It then follows that

$$\lambda_n^\downarrow(A) - \lambda_n^\uparrow(B) \leq \Phi_1(A) - \Phi_2(B) \leq \lambda_1^\downarrow(A) - \lambda_1^\uparrow(B).$$

Therefore,

$$-k \leq \Phi_1(A) - \Phi_2(B) \leq k$$

where

$$k = \max \left\{ \left| \lambda_n^\downarrow(A) - \lambda_n^\uparrow(B) \right|, \left| \lambda_1^\downarrow(A) - \lambda_1^\uparrow(B) \right| \right\}$$

Further, if  $X$  is Hermitian and  $\pm X \leq kI$  then  $\|X\| \leq k$ , and therefore

$$\|\Phi_1(A) - \Phi_2(B)\| \leq k.$$

The assertions of the theorem now follow from the fact that

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| = \max_j \left| \lambda_j^\downarrow(A) - \lambda_j^\uparrow(B) \right| = k. \quad \blacksquare$$

We note that the inequality (2.4) and the second inequality (1.4) are independent. The maps

$$\Phi_1(A) = \frac{1}{n-1}(tr A - A) \text{ and } \Phi_2(B) = B$$

are positive unital linear maps. For these maps, the inequality (2.4) becomes

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \geq \frac{1}{n-1} \left\| A - B + n \left( B - \frac{tr A}{n} \right) \right\|. \quad (2.5)$$

For  $A = B$ , the inequality (2.5) gives

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \geq \frac{n}{n-1} \left\| A - \frac{tr A}{n} \right\|, \quad (2.6)$$

while Weyl's inequality (1.4) gives  $\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \geq 0$ .

But for  $B = \frac{tr A}{n}I$ , we respectively have from (2.5) and (1.4),

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \geq \frac{1}{n-1} \|A - B\|$$

and

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \geq \|A - B\|.$$

Choosing different linear maps in Theorem 2.2 and Corollary 2.1, we can obtain various interesting inequalities which provide lower bounds for the maximum distance between eigenvalues of two normal matrices. We demonstrate some special cases here.

Choose  $\varphi_1(A) = a_{ii}$  and  $\varphi_2(B) = b_{jj}$  in (2.3), we have

$$\max_{i,j} |\lambda_i(A) - \lambda_j(B)| \geq \max_{i,j} |a_{ii} - b_{jj}|. \quad (2.7)$$

Let  $D$  be the diagonal part of  $A$ . From (2.7) we have

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \max_{i,j} |a_{ii} - a_{jj}|. \quad (2.8)$$

Note that (2.8) provides a refinement of the inequality  $\text{spd}(A) \geq \max_{i,j} |a_{ii} - a_{jj}|$ . By

using spectral theorem,  $a_{ii} = \sum_{i=1}^n \lambda_i(A) p_i$  where  $p_i$  are non-negative real numbers such

that  $\sum_{i=1}^n p_i = 1$ , therefore

$$\begin{aligned} |\lambda_i(A) - a_{jj}| &= \left| \sum_{j=1}^n p_j (\lambda_i(A) - \lambda_j(A)) \right| \leq \sum_{j=1}^n p_j |\lambda_i(A) - \lambda_j(A)| \\ &\leq \max_{i,j} |\lambda_i(A) - \lambda_j(A)| = \text{spd}(A). \end{aligned}$$

So  $\text{spd}(A) \geq \max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \max_{i,j} |a_{ii} - a_{jj}|$ .

Let  $A = D + N$ . If  $N$  is also a normal matrix, as in case of circulant and Hermitian matrices, then

$$\max_{i,j} |\lambda_i(A) - \lambda_j(N)| \geq \max_i |a_{ii}|.$$

Let  $B = \frac{A+A^*}{2}$  and  $C = \frac{A-A^*}{2i}$ . If  $A$  is normal,  $\text{Re}(\lambda_i(A)) = \lambda_i(B)$  and  $\text{Im}(\lambda_i(A)) = \lambda_i(C)$ . We therefore have

$$\max_{i,j} |\lambda_i(A) - \text{Re}(\lambda_j(A))| \geq \max_{i,j} |a_{ii} - \text{Re} a_{jj}|,$$

$$\max_{i,j} |\lambda_i(A) - \text{Im}(\lambda_j(A))| \geq \max_{i,j} |a_{ii} - \text{Im} a_{jj}|.$$

For arbitrary matrices, we have

$$\max_{i,j} \left| \lambda_i \left( \frac{A+A^*}{2} \right) - \lambda_j \left( \frac{A-A^*}{2i} \right) \right| \geq \max_{i,j} |\text{Re} a_{ii} - \text{Im} a_{jj}|$$

Note that if  $A$  is normal,  $\lambda_i \left( \frac{A+A^*}{2} \right) - \lambda_j \left( \frac{A-A^*}{2i} \right) = \text{Re}(\lambda_i(A)) - \text{Im}(\lambda_j(A))$ .

We now obtain some more inequalities in the following corollaries.

**Corollary 2.2.** Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be Hermitian matrices. Then

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(B)\| \geq \frac{1}{2} \left| \alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4|a_{ij} + b_{ij}|^2} \right| \quad (2.9)$$

where  $\alpha = a_{ii} - b_{jj}$ ,  $\beta = a_{jj} - b_{ii}$  and  $i \neq j$ .

**Proof.** The maps

$$\Phi_1(A) = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \quad \text{and} \quad \Phi_2(B) = \begin{bmatrix} b_{jj} & -b_{ij} \\ -b_{ji} & b_{ii} \end{bmatrix}.$$

are positive unital linear maps, and

$$\Phi_1(A) - \Phi_2(B) = \begin{bmatrix} a_{ii} - b_{jj} & a_{ij} + b_{ij} \\ a_{ji} + b_{ji} & a_{jj} - b_{ii} \end{bmatrix}.$$

is a Hermitian matrix with eigenvalues

$$\frac{1}{2} \left( \alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4|a_{ij} + b_{ij}|^2} \right).$$

So  $\|\Phi_1(A) - \Phi_2(B)\| = \frac{1}{2} \left| \alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4|a_{ij} + b_{ij}|^2} \right|$ . The inequality (2.9) now

follows from Theorem 2.2. ■

In the special case when  $A = B$ , inequality (2.9) gives Mirsky bound [11] for the spread of  $A$ . It also follows from (2.9) that

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(D)\| \geq \sqrt{(a_{ii} - a_{jj})^2 + |a_{ij}|^2}$$

and

$$\|\text{Ei } g^\downarrow(A) - \text{Ei } g^\uparrow(N)\| \geq \frac{1}{2} \left| a_{ii} + a_{jj} \pm \sqrt{(a_{ii} - a_{jj})^2 + 16 |a_{ij}|^2} \right|.$$

**Corollary 2.3.** Let  $A$  and  $B$  be normal matrices. Let  $I$  and  $J$  be any two subsets of  $\{1, 2, \dots, n\}$  and let  $|I|$  and  $|J|$  denote the cardinality of  $I$  and  $J$ . Then

$$\max_{i,j} |\lambda_i(A) - \lambda_j(B)| \geq \max_{i,j} \left| \frac{1}{|I|} \sum_{i,j \in I} a_{ij} - \frac{1}{|J|} \sum_{i,j \in J} b_{ij} \right|. \quad (2.10)$$

**Proof.** Choose

$$\varphi_1(A) = \frac{1}{|I|} \sum_{i,j \in I} a_{ij} \text{ and } \varphi_2(B) = \frac{1}{|J|} \sum_{i,j \in J} b_{ij}$$

and use (2.3), we immediately get (2.10). ■

A special case of Corollary 2.3 when  $A = B$  is Theorem 2.2 of Johnson et al [8]. The inequality (2.10) for  $B = D$  gives

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \max_{i,j} \left| \frac{1}{|I|} \sum_{i,j \in I} a_{ij} - \frac{1}{|J|} \sum_{i \in J} a_{ii} \right|.$$

It also follows from (2.10) that

$$\max_{i,j} |\lambda_i(A) - \text{Re}(\lambda_j(A))| \geq \max_{i,j} \left| \frac{1}{|I|} \sum_{i,j \in I} a_{ij} - \frac{1}{|J|} \sum_{i,j \in J} \text{Re } a_{ij} \right|,$$

$$\max_{i,j} |\lambda_i(A) - \text{Im}(\lambda_j(A))| \geq \max_{i,j} \left| \frac{1}{|I|} \sum_{i,j \in I} a_{ij} - \frac{1}{|J|} \sum_{i,j \in J} \text{Im } a_{ij} \right|$$

and

$$\max_{i,j} |\text{Re}(\lambda_i(A)) - \text{Im}(\lambda_j(A))| \geq \max_{i,j} \left| \frac{1}{|I|} \sum_{i,j \in I} \text{Re } a_{ij} - \frac{1}{|J|} \sum_{i,j \in J} \text{Im } a_{ij} \right|.$$

**Corollary 2.4.** Let  $A$  and  $B$  be normal matrices. Then

$$\max_{i,j} |\lambda_i(A) - \lambda_j(B)| \geq \left| \frac{1}{n-1} \sum_{i \neq j} a_{ij} + \frac{1}{n} \sum_{i,j} (b_{ij} - a_{ij}) \right|. \quad (2.11)$$



**Proof.** For the positive unital linear functionals

$$\varphi_1(A) = \frac{1}{n} \left( \text{tr} A - \frac{1}{n-1} \sum_{i \neq j} a_{ij} \right), \quad \varphi_2(B) = \frac{1}{n} \sum_{i,j} b_{ij} \quad \text{and} \quad \varphi_3(A) = \frac{1}{n} \sum_{i,j} a_{ij},$$

we have

$$\begin{aligned} |\varphi_1(A) - \varphi_2(B)| &= |\varphi_1(A) - \varphi_3(A) + \varphi_3(A) - \varphi_2(B)| \\ &= \left| \frac{1}{n-1} \sum_{i \neq j} a_{ij} + \frac{1}{n} \sum_{i,j} (b_{ij} - a_{ij}) \right|. \end{aligned}$$

The assertions of the corollary now follows from the inequality (2.3). ■

Theorem 5 of Merikoski and Kumar [9] is a special case of our Corollary 2.4,  $A = B$ .

For  $A = D$  and  $B = A$ ,

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \frac{1}{n} \left| \sum_{i \neq j} a_{ij} \right|.$$

and

$$\max_{i,j} |\text{Re}(\lambda_i(A)) - \text{Im}(\lambda_j(A))| \geq \left| \frac{1}{n-1} \sum_{i \neq j} \text{Re} a_{ij} + \frac{1}{n} \sum_{i,j} (\text{Im} a_{ij} - \text{Re} a_{ij}) \right|.$$

**Corollary 2.5.** Let  $A$  and  $B$  be normal matrices. Then

$$\max_{i,j} |\lambda_i(A) - \lambda_j(B)| \geq \frac{1}{2} |(\alpha + \beta) + (a_{ij}e^{i\theta} + a_{ji}e^{-i\theta})|. \quad (2.12)$$

**Proof.** Let

$$\varphi_1(A) = \frac{1}{2} (a_{ii} + a_{jj} + a_{ij}e^{i\theta} + a_{ji}e^{-i\theta}) \quad \text{and} \quad \varphi_2(B) = \frac{1}{2} (b_{ii} + b_{jj}).$$

$\varphi_1(A)$  and  $\varphi_2(B)$  are positive unital linear functionals, see [5]. The inequality (2.12) follows from (2.3). ■

Let  $B = D$ . Then, (2.12) gives

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \frac{1}{2} \max_{i \neq j} |a_{ij}e^{i\theta} + a_{ji}e^{-i\theta}|.$$

So,

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \frac{1}{2} \max_{i \neq j} (|a_{ij}| + |a_{ji}|).$$

If  $A$  is Hermitian, we have

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \max_{i \neq j} |a_{ij}|.$$

Likewise, we can see that

$$\max_{i,j} |\lambda_i(A) - \lambda_j(D)| \geq \max_{p \neq q} \left| \frac{1}{n} \sum_{i,j} a_{ij} - \frac{a_{pp} + a_{qq}}{2} \right|.$$

### 3 Bounds for spread

The spread of  $A$ , denoted by  $\text{spd}(A)$ , is defined as

$$\text{spd}(A) = \max_{1 \leq i, j \leq n} |\lambda_i(A) - \lambda_j(A)|,$$

where  $\lambda_1(A), \dots, \lambda_n(A)$  are the eigenvalues of  $A$ . Beginning with Mirsky [10] several authors have worked on the bounds for spread of matrices, see [3, 4, 5, 6] and references therein. We mention here some lower bounds for the spread related to perturbation bounds. It is clear from (2.6) that for any Hermitian element  $A \in \mathbb{M}(n)$ , we have

$$\text{spd}(A) \geq \frac{n}{n-1} \left\| A - \frac{\text{tr} A}{n} \right\|$$

We prove that this inequality also holds for normal matrices.

**Theorem 3.1.** For any normal matrix  $A$ , we have

$$\text{spd}(A) \geq \frac{n}{n-1} \left\| A - \frac{\text{tr} A}{n} \right\|. \quad (3.1)$$

**Proof.** It is immediate that

$$\left| \lambda_j - \frac{\text{tr} A}{n} \right| \leq \frac{1}{n} \sum_{i=1, i \neq j}^n |\lambda_i - \lambda_j| \leq \frac{n-1}{n} \max_i |\lambda_i - \lambda_j|,$$

for all  $j = 1, 2, \dots, n$ . So

$$\text{spd}(A) \geq \frac{n}{n-1} \max_j \left| \lambda_j - \frac{\text{tr} A}{n} \right|. \quad (3.2)$$

For a normal matrix  $A$ , we have

$$\max_j \left| \lambda_j - \frac{\text{tr} A}{n} \right| = \left\| A - \frac{\text{tr} A}{n} \right\|. \quad (3.3)$$

Combining (3.2) and (3.3), we immediately get (3.1). ■

It may be noted here that for a normal matrix  $A$ ,

$$\max_j \left| \lambda_j - \frac{\text{tr} A}{n} \right| \geq \left\| \varphi \left( A - \frac{\text{tr} A}{n} \right) \right\|.$$

We therefore also have

$$\text{spd}(A) \geq \frac{n}{n-1} \left\| \varphi(A) - \frac{\text{tr} A}{n} \right\|. \quad (3.4)$$

Choose  $\varphi(A) = \frac{1}{n} \sum_{i,j=1}^n a_{ij}$ , (3.4) gives

$$\text{spd}(A) \geq \frac{1}{n-1} \left| \sum_{i \neq j} a_{ij} \right|.$$

This is Theorem 2.1 of Johnson et al [8] and Theorem 5 of Merikoski and Kumar [9]. We now prove a refinement of Theorem 5 in [9].

**Theorem 3.2.** Let  $A$  be a normal matrix. Then

$$\text{spd}(A) \geq \max_{i,j} \left| \lambda_i(A) - \frac{1}{n-1} \sum_{k \neq j, k=1}^n \lambda_k(A) \right| \geq \frac{1}{n-1} \left| \sum_{i \neq j} a_{ij} \right|. \quad (3.5)$$

**Proof.** To prove first inequality (3.5), note that

$$\begin{aligned} \left| \lambda_i(A) - \frac{1}{n-1} \sum_{k \neq j, k=1}^n \lambda_k(A) \right| &= \frac{1}{n-1} \left| \sum_{k \neq j, k=1}^n \lambda_i(A) - \lambda_k(A) \right| \\ &\leq \frac{1}{n-1} \sum_{k \neq j, k=1}^n |\lambda_i(A) - \lambda_k(A)| \\ &\leq \max_j |\lambda_i(A) - \lambda_j(A)|. \end{aligned}$$

For  $B = \frac{1}{n-1} (\text{tr} A - A)$  and  $\varphi_1 = \varphi_2$ , the inequality (2.3) gives

$$\max_{i,j} |\lambda_i(A) - \lambda_j(B)| \geq \frac{n}{n-1} \left| \varphi_1(A) - \frac{\text{tr} A}{n} \right|. \quad (3.6)$$

The eigenvalues of  $B$  are  $\frac{1}{n-1} \sum_{k \neq j, k=1}^n \lambda_k(A)$ ,  $j = 1, 2, \dots, n$ . Choose  $\varphi_1(A) = \frac{1}{n} \sum_{i,j} a_{ij}$  in (3.6), we immediately get the second inequality in (3.5). ■

Analogous bound for the ratio spread  $\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)$  of a positive definite matrix is proved in the following theorem.

**Theorem 3.3.** Let  $\Phi : \mathbb{M}(n) \rightarrow \mathbb{M}(k)$  be a positive unital linear map. Let  $A \in \mathbb{M}(n)$  be a positive definite matrix. For  $m \leq A \leq M$ , we have

$$\left(\frac{m}{M}\right)^{\frac{n-1}{n}} \leq (\det A)^{\frac{-1}{n}} \Phi(A) \leq \left(\frac{M}{m}\right)^{\frac{n-1}{n}}. \quad (3.7)$$

**Proof.** Let  $\lambda_i$  be the eigenvalues of  $A$ . It is clear that  $m \leq \lambda_i \leq M$ ,  $i = 1, 2, \dots, n$ . Therefore,

$$m^{n-1}M \leq \lambda_1 \lambda_2 \dots \lambda_n \leq M^{n-1}m.$$

So,

$$m^{\frac{n-1}{n}} M^{\frac{1}{n}} \leq (\det A)^{\frac{1}{n}} \leq M^{\frac{n-1}{n}} m^{\frac{1}{n}}$$

and therefore

$$\frac{1}{M^{\frac{n-1}{n}} m^{\frac{1}{n}}} \leq \frac{1}{(\det A)^{\frac{1}{n}}} \leq \frac{1}{m^{\frac{n-1}{n}} M^{\frac{1}{n}}}. \quad (3.8)$$

Also,  $m \leq A \leq M$  therefore

$$m \leq \Phi(A) \leq M. \quad (3.9)$$

The inequality (3.7) now follows from (3.8) and (3.9). ■

Let  $\Phi : \mathbb{M}(n) \rightarrow \mathbb{M}(k)$  be a positive unital linear map. Let  $A$  be a Hermitian element of  $\mathbb{M}(n)$  such that  $mI \leq A \leq MI$ . Bhatia and Davis [3] have proved that

$$\Phi(A^2) - \Phi(A)^2 \leq (M - \Phi(A))(\Phi(A) - m). \quad (3.10)$$

We use (3.10), and obtain a refinement of (1.1) for positive definite matrices under a certain condition.

**Theorem 3.4.** Let  $\Phi : \mathbb{M}(n) \rightarrow \mathbb{M}(k)$  be a positive unital linear map and  $A$  be any positive semidefinite matrix in  $\mathbb{M}(n)$  such that  $mI \leq A \leq MI$ . If  $\Phi(A^2) \geq 2\Phi(A)^2$  and  $\Phi(A) > 0$ , then

$$\Phi(A^2) \leq (M - m)\Phi(A). \quad (3.11)$$

**Proof.** From (3.10), we have

$$\Phi(A^2) \leq (m + M - mM\Phi(A)^{-1})\Phi(A). \quad (3.12)$$

If  $\Phi(A^2) \geq 2\Phi(A)^2$ , then

$$\Phi(A) \leq \sqrt{\Phi(A^2) - \Phi(A)^2}. \quad (3.13)$$

From (1.1) and (3.13), we get that

$$\Phi(A) \leq \frac{M - m}{2}.$$

Therefore

$$m + M - mM\Phi(A)^{-1} \leq m + M - \frac{2mM}{M - m} \leq M - m.$$

Since  $\Phi(A)$  and  $\Phi(A)^{-1}$  commute, therefore

$$(m + M - mM\Phi(A)^{-1})\Phi(A) \leq (M - m)\Phi(A). \quad (3.14)$$

From (3.12) and (3.14), we get (3.11). ■

We now show that above theorem provides a refinement of the inequality (1.1) for linear functionals.

**Corollary 3.1.** Let  $\varphi : \mathbb{M}(n) \rightarrow \mathbb{C}$  be a positive unital linear functional and  $A$  be any positive semidefinite matrix in  $\mathbb{M}(n)$  such that  $mI \leq A \leq MI$ . If  $\varphi(A^2) \geq 2\varphi(A)^2$  and  $\varphi(A) > 0$ , then

$$\varphi(A^2) - \varphi(A)^2 \leq \left( \frac{\varphi(A^2)}{2\varphi(A)} \right)^2 \leq \frac{(M - m)^2}{4}. \quad (3.15)$$

**Proof.** The second inequality in (3.15) follows from inequality (3.11). The first inequality holds if and only if

$$4\varphi(A^2)\varphi(A)^2 - 4\varphi(A)^4 \leq \varphi(A^2)^2$$

if and only if

$$(\varphi(A^2) - 2\varphi(A)^2)^2 \geq 0$$

This is true. ■

**Example.** Let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 9 & 9 & 5 \\ 9 & 9 & 5 \\ 5 & 5 & 3 \end{bmatrix}$$

Let  $\varphi(A) = a_{ii}$ , then  $\varphi(A^2) \geq 2\varphi(A)^2$ . From (1.1) we have  $M - m \geq 4.4721$ . From (3.15),  $M - m \geq 4.5$ . Note that  $M - m \geq 4.5616$ .

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